Prove that $g(x)=\frac{1}{2}\ln\frac{1+x}{1-x}$ is the inverse of $f(x)=\tanh x$ by simplifying $f(g(x))$. **SCORE: ______ / 5 PTS**

<u>You may need to use an exponential definition of tanh x.</u>

$$\frac{e^{2\left(\frac{1}{2}\ln\frac{1+x}{1-x}\right)}-1}{e^{2\left(\frac{1}{2}\ln\frac{1+x}{1-x}\right)}+1}$$

$$=\frac{e^{\ln\frac{1+x}{1-x}}-1}{e^{\ln\frac{1+x}{1-x}}+1} \quad (1)$$

$$=\frac{\frac{1+x}{1-x}-1}{\frac{1+x}{1-x}+1}\cdot\frac{1-x}{1-x} \quad (1) \quad =\frac{1+x-(1-x)}{1+x+(1-x)} \quad (1) \quad =\frac{2x}{2} \quad (1) \quad = x \quad (1)$$

☆ ALSO SEE ALTERNATE SOLUTION BELOW

GRADE AGAINST 1 VERSION OF SOLUTION ONLY

Prove that $g(x)=\frac{1}{2}\ln\frac{1+x}{1-x}$ is the inverse of $f(x)=\tanh x$ by simplifying $f(g(x))$. **SCORE: ______ / 5 PTS**

You may need to use an exponential definition of $\tanh x$.

$$\frac{e^{\frac{1}{2}\ln\frac{1+x}{1-x}}-e^{-\frac{1}{2}\ln\frac{1+x}{1-x}}}{e^{\frac{1}{2}\ln\frac{1+x}{1-x}}+e^{-\frac{1}{2}\ln\frac{1+x}{1-x}}} \quad (\tfrac{1}{2})$$

$$=\frac{\sqrt{\frac{1+x}{1-x}}-\frac{1}{\sqrt{\frac{1+x}{1-x}}}}{\sqrt{\frac{1+x}{1-x}}+\frac{1}{\sqrt{\frac{1+x}{1-x}}}} \quad (\tfrac{1}{2}) \cdot \frac{\sqrt{\frac{1+x}{1-x}}}{\sqrt{\frac{1+x}{1-x}}}$$

$$=(1)\ \frac{\frac{1+x}{1-x}-1}{\frac{1+x}{1-x}+1}\cdot\frac{1-x}{1-x}=\overset{(1)}{\frac{1+x-(1-x)}{1+x+(1-x)}}=\overset{(1)}{\frac{2x}{2}}=\overset{(1)}{x}$$

Rewrite $\coth(\frac{1}{2}\ln 5)$ in terms of exponential functions and simplify. **SCORE: ______ / 3 PTS**

$$\frac{e^{\frac{1}{2}\ln 5}+e^{-\frac{1}{2}\ln 5}}{e^{\frac{1}{2}\ln 5}-e^{-\frac{1}{2}\ln 5}} = \frac{\sqrt{5}+\frac{1}{\sqrt{5}}}{\sqrt{5}-\frac{1}{\sqrt{5}}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{5+1}{5-1} = \frac{6}{4} = \frac{3}{2}$$

(1) $(\frac{1}{2})$ (1) $(\frac{1}{2})$

OR $$\frac{e^{2(\frac{1}{2}\ln 5)}+1}{e^{2(\frac{1}{2}\ln 5)}-1} = \frac{e^{\ln 5}+1}{e^{\ln 5}-1} = \frac{5+1}{5-1} = \frac{6}{4} = \frac{3}{2}$$

$(1\frac{1}{2})$ (1) $(\frac{1}{2})$

★ GRADE AGAINST 1 VERSION OF SOLUTION ONLY

Write and **prove** a formula for $\sinh(x-y)$ in terms of $\sinh x$, $\sinh y$, $\cosh x$ and $\cosh y$. **SCORE: ______ / 5 PTS**

$\sinh x \cosh y - \cosh x \sinh y$ (1)

$= \frac{e^x - e^{-x}}{2} \frac{e^y + e^{-y}}{2} - \frac{e^x + e^{-x}}{2} \frac{e^y - e^{-y}}{2}$ (1)

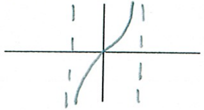
(1) $= \frac{e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y} - (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})}{4}$ (1) ($\frac{1}{2}$)

$= \frac{2e^{x-y} - 2e^{-x+y}}{4}$

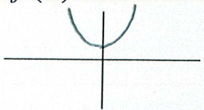
$= \frac{e^{x-y} - e^{-(x-y)}}{2}$ ($\frac{1}{2}$) $= \sinh(x-y)$

Sketch the general shape and position (including asymptotes) of the following graphs. **SCORE: ______ / 3 PTS**
(Don't worry about specific $x-$ or $y-$ coordinates.)

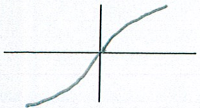
$f(x) = \tanh^{-1} x$



$f(x) = \cosh x$



$f(x) = \sinh^{-1} x$



GRADED BY ME

There is an identity involving $\sinh x$ and $\cosh x$ that resembles a Pythagorean identity from trigonometry. **SCORE: ______ / 8 PTS**

[a] Write that identity involving $\sinh x$ and $\cosh x$. **You do NOT need to prove the identity.**

(1) $\cosh^2 x - \sinh^2 x = 1$

[b] Use the answer of [a] to find and **prove** an identity involving $\operatorname{sech}^2 x$ that resembles a Pythagorean identity from trigonometry.

(1) $\dfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \dfrac{1}{\cosh^2 x}$ $\longrightarrow$ $1 - \tanh^2 x = \operatorname{sech}^2 x$ (1)

[c] Write the identity for $\cosh 2x$ that uses both $\sinh x$ and $\cosh x$ simultaneously. **You do NOT need to prove the identity.**

(1) $\cosh 2x = \cosh^2 x + \sinh^2 x$

[d] Use the answers of [a] and [c] to find and **prove** an identity for $\cosh 2x$ that uses only $\cosh x$.

$\sinh^2 x = \cosh^2 x - 1$

(1) $\cosh 2x = \cosh^2 x + (\cosh^2 x - 1) = 2\cosh^2 x - 1$ ($\frac{1}{2}$)

[e] If $\coth x = -\frac{5}{3}$, find $\operatorname{sech} x$ using identities.

You must explicitly show the use of the identities but you do NOT need to prove the identities.
Do NOT use inverse hyperbolic functions nor their logarithmic formulae in your solution.

$\tanh x = \dfrac{1}{\coth x} = -\dfrac{3}{5}$ ($\frac{1}{2}$)

$1 - \left(-\frac{3}{5}\right)^2 = \operatorname{sech}^2 x$ ($\frac{1}{2}$)

$1 - \frac{9}{25} = \operatorname{sech}^2 x$

$\frac{16}{25} = \operatorname{sech}^2 x$ ($\frac{1}{2}$)

$\operatorname{sech} x = \frac{4}{5}$ ($\frac{1}{2}$) (SINCE $\operatorname{sech} x = \dfrac{1}{\cosh x} > 0$ FOR ALL x) ($\frac{1}{2}$)